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BROWN-GITLER SPECTRA AT BP⟨2⟩

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Brown-Gitler spectra for the homology theory associated to BP⟨2⟩ are constructed. Complexes adapted to the new Brown-Gitler spectra are constructed and a spectral sequence converging to stable maps into these spectra is examined.

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Brown-Gitler spectra BP⟨2⟩

In [1], Brown and Gitler constructed certain spectra $B(k)$ with the property that $H^*B(k)\cong A/I_k$ for certain ideals I_k over the Steenrod algebra A. In [6], Goerss, Jones and Mahowald constructed what they called generalized Brown-Gitler spectra at BP⟨1⟩, written $\mathrm{BP}\langle 1\rangle^k$, with the property that $H^*\mathrm{BP}\langle 1\rangle^k\cong A/I_k+J_1$ where J_1 is the ideal with the property that $H^*\mathrm{BP}\langle 1\rangle\cong A/J_1$. This paper is devoted to constructing Brown-Gitler spectra at BP⟨2⟩ with $H^*\mathrm{BP}\langle 2\rangle^k\cong A/I_k+J_2$ where $H^*\mathrm{BP}\langle 2\rangle\cong A/J_2$.

Although the original Brown-Gitler spectra have been put to a variety of uses in homotopy theory, the applications of generalized Brown-Gitler spectra have been restricted to splittings of $\mathrm{bo}\wedge\mathrm{bo}$ (see [9]), $\mathrm{BP}\langle 1\rangle\wedge\mathrm{BP}\langle 1\rangle$ (see [7]), and $\mathrm{BP}\langle 1\rangle\wedge\mathrm{BP}\langle n\rangle$ (see [8]). In fact if J_n is an ideal satisfying $H^*\mathrm{BP}\langle n\rangle\cong A/J_n$ (one should interpret $\mathrm{BP}\langle -1\rangle$ as $H\mathbb{Z}/p$, $\mathrm{BP}\langle 0\rangle$ as $H\mathbb{Z}_p^\wedge$, and J_{-1} as the empty set), then $H^*(\mathrm{BP}\langle n\rangle\wedge\mathrm{BP}\langle n\rangle)\cong\bigoplus_{k\geqslant 0}\Sigma^{2(p-1)k}H^*\mathrm{BP}\langle n\rangle\otimes A/I_k+J_n$ and one hopes to be able to realize A/I_k+J_n as the mod p cohomology of a spectrum $\mathrm{BP}\langle n\rangle^k$ and then realize this as a splitting of spectra. This is the motivation for constructing $\mathrm{BP}\langle 2\rangle^k$.

Having fixed k, the first task is to construct an acyclic resolution of A/I_k+J_n. $A/I_k+J_n\leftarrow D_0^n\rightarrow D_1^n\leftarrow D_2^n\leftarrow\cdots$. Brown and Gitler [1] used the lambda algebra to produce the corresponding resolution; Goerss, Jones and Mahowald [6] used a modification of the lambda algebra. The appropriate modification for BP⟨2⟩ is apparent in [6] and little will be said here about the details needed to get the acyclic resolution,

$$A/I_k+J_2\leftarrow D_0^2\leftarrow D_1^2\leftarrow D_2^2\leftarrow\cdots.$$

Along with this, one wants to have the resolution realized by maps of spectra,

$$K_0^n \xrightarrow{d_0} \Sigma K_1^n \xrightarrow{d_1} \Sigma^2 K_2^n \to \cdots.$$

Since in [1] the resolution was free over the Steenrod algebra, they got the maps of spectra for free. This was not the case in either [6] or this paper. In our resolution the first three terms are direct sums of suspensions of $H^*\mathrm{BP}\langle 2\rangle$, $H^*\mathrm{BP}\langle 1\rangle$ and $H^*H\mathbb{Z}_p^\wedge$ respectively. The only new constructions needed are maps $S^j:\mathrm{BP}\langle 2\rangle \to \Sigma^{2(p-1)}\mathrm{BP}\langle 1\rangle$ with the property that in cohomology $S^{j*}(1)=\chi(P^j)$, where 1 and $\chi(P^j)$ represent their respective equivalence classes in A/J_1 and A/J_2. This is a consequence of the fact that the Adams spectral sequence for $[\mathrm{BP}\langle 2\rangle, \mathrm{BP}\langle 1\rangle]$ collapses to its E_2 term, as proved in [8].

Conceptually, the next step would be to construct a tower of spectra with the following properties.

Theorem A. *For any n between* -1 *and* 2 *there is a tower of spectra*

$$X_{q+1} \xrightarrow{p_1} X_q \to \cdots \to X_1 \xrightarrow{p_0} X_0$$

with the property that

(a) *there are maps* $e_1: X_q \to \Sigma K_{q+1}^n$ *where* K_{q+1}^n *are the spectra mentioned in the previous paragraph,*

(b) $X_0 = K_0^n = \mathrm{BP}\langle n\rangle$, $e_0 = d_0$,

(c) $$K_{q+1}^n \xrightarrow{i_{q+1}} X_{q+1} \xrightarrow{p_q} X_q \xrightarrow{e_q} \Sigma K_{q+1}^n,$$

is a cofiber sequence (defining i_{q+1}*),*

(d) $e_q i_q = d_q$, $i_0 = Id$,

(e) *For any CW complex Z, the induced map of homology theories* $e_q^*: (X_q)_m Z \to (K_{q+1}^n)_m Z$ *is zero for* $m \leqslant 2p(k+1)-1$ *and* $q \geqslant n$.

This was proved in [6] for $-1 \leqslant n \leqslant 1$ (one should identify $\mathrm{BP}\langle -1\rangle$ with $H\mathbb{Z}/p$ and $\mathrm{BP}\langle 0\rangle$ with $H\mathbb{Z}_p^\wedge$). Given Theorem A, the $\mathrm{BP}\langle 2\rangle^k$ are defined to be the homotopy inverse limit of the X_q tower in the theorem when $n=2$.

Theorem B. *For each* $k \geqslant 0$ *and prime p, there is a p-complete spectrum* $\mathrm{BP}\langle 2\rangle^k$ *and a map* $w: \mathrm{BP}\langle 2\rangle^k \to \mathrm{BP}\langle 2\rangle$ *such that*

(a) $H^*\mathrm{BP}\langle 2\rangle^k \cong A/A\{\beta, \beta P^1 \beta P^{p+1}, \chi P^j \mid j>k\} \cong A/I_k + J_2$

(b) $\{\mathrm{Im}\, w_*: \mathrm{BP}\langle 2\rangle_m^k Z \to \mathrm{BP}\langle 2\rangle_m Z\} \subseteq \bigcap_{j>k} \ker S_*^j$.

If Z is a CW complex with $H\mathbb{Z}_{p*}^\wedge Z$ *an* $\mathbb{F}_p$ *vector space and* $m \leqslant 2p(k+1)-1$ *then* $\mathrm{Im}\, w_*$ *is all of* $\bigcap_{j>k} \ker S_*^j$.

Partial Proof. To prove part (a), apply H^* to the tower of fibrations in Theorem A. The spectral sequence associated to the unravelled exact couple converges to

$H^*\mathrm{holim}_q X_q = H^*\mathrm{BP}\langle 2\rangle^k$ and has $E^1_{s,*} \cong H^*K^2_s \cong D^2_s$. Since $0 \leftarrow A/I_k + J_2 \leftarrow D^2_0 \leftarrow D^2_1 \leftarrow \cdots$ was an acyclic resolution, the E^2 term of this spectral sequence has $E^2_0 = \mathrm{Coker}\, H^*d_0$ and $E^2_i = 0$ for $i \geq 1$. Thus $H^*\mathrm{BP}\langle 2\rangle^k$ is as claimed in (a).

The map $w: \mathrm{BP}\langle 2\rangle^k \to \mathrm{BP}\langle 2\rangle$ is just

$$\mathrm{BP}\langle 2\rangle^k = \mathrm{holim}_q X_q \to X_0 = \mathrm{BP}\langle 2\rangle$$

which can be factored as

$$\mathrm{BP}\langle 2\rangle^k \to X_1 \xrightarrow{p_0} \mathrm{BP}\langle 2\rangle.$$

Since

$$X_1 \to \mathrm{BP}\langle 2\rangle \xrightarrow{\bigcap_{j>k} S^j} \bigcup_{j>k} \Sigma^{2(p-1)j}\, \mathrm{BP}\langle 1\rangle = \Sigma K^2_1$$

is a cofibration we get the first part of assertion (b). The proof of the last part of (b) requires some knowledge of how Theorem A is proved and will appear in Section 3.

Finally taking the tower of fibrations created in Theorem A, smashing with any spectrum Y and taking stable homotopy gives a spectral sequence converging to $\mathrm{BP}\langle n\rangle^k_* Y$ with

$$E^1_{s,t}(Y \wedge \mathrm{BP}\langle n\rangle) \cong (K^n_s)_{t-s} Y = \pi_{t-s}(K_s \wedge Y).$$

When $n = 0$ or -1 and Y is the suspension spectrum of a space then part (e) of Theorem A implies that

$$E^1_{s,t}(Y \wedge \mathrm{BP}\langle n\rangle) \cong E^\infty_{s,t}(Y \wedge \mathrm{BP}\langle n\rangle) \quad \text{for } t - s \leq 2p(k+1) - 2.$$

When $n = 1$ this becomes $E^2_{s,t}(Y \wedge \mathrm{BP}\langle n\rangle) \cong E^\infty_{s,t}(Y \wedge \mathrm{BP}\langle n\rangle)$ for $t - s \leq 2p(k+1) - 2$. For Brown-Gitler spectra at $\mathrm{BP}\langle 2\rangle$ we get

Corollary C. *If Y is the suspension spectrum of a space then in the spectral sequence converging to* $\mathrm{BP}\langle 2\rangle^k_* Y$,

$$E^3_{s,t}(Y \wedge \mathrm{BP}\langle 2\rangle) \cong E^\infty_{s,t}(Y \wedge \mathrm{BP}\langle 2\rangle) \quad \textit{for all } t - s \leq 2p(k+1) - 2.$$

The dual to this situation involves applying the functor $[Y, \;]$ to the tower of Theorem A to get a spectral sequence converging to $[Y, \mathrm{BP}\langle n\rangle^k]^{s-t}$ with

$$E^{s,t}_1(Y, \mathrm{BP}\langle n\rangle) \cong [Y, K^n_s]^{s-t}.$$

In order to get the collapsing result analogous to Corollary C, [6] introduced the notion of spacelike spectra.

Definition. Let Y be a finite CW spectrum. Y is said to be *spacelike of dimension* n if there is a spectrum T and a map $f: T \to Y$ so that

(i) $T = \Sigma^n DZ$ where Z is a finite CW complex, and
(ii) $f^*: H^*Y \to H^*T$ is injective.

Corollary D. *If Y is spacelike of dimension n, then in the spectral sequence converging to* $[Y, \mathrm{BP}\langle 2\rangle^k]^{s-t}$, *for all* $s-t \geq n-2p(k+1)+2$

$$E_3^{s,t} \cong E_\infty^{s,t} \quad \textit{and} \quad E_1^{s,t} \cong E_\infty^{s,t} \quad \textit{for } s \geq 3.$$

Although conceptually the construction of the X_q tower comes after realizing the D_n^2 resolution, performing the construction requires the use of some complicated machinery and while constructing the tower one must simultaneously construct maps from the spectra needed for the $\mathrm{BP}\langle 2\rangle^k$ construction to those that were used in the construction of the original Brown-Gitler spectra.

In Section 1 the acyclic resolution of $M(2,k)$ is given and realized. Section 2 contains the definition and construction of adapted complexes, the complicated machinery referred to in the previous paragraph. Section 3 gives the proof of Theorem A and Theorem B while Section 4 concludes with a proof of Corollary D.

Throughout this paper we are working with a fixed prime p and a fixed positive integer k. Homology is always with mod p coefficients unless otherwise specified.

In this paper, references to earlier results about Brown-Gitler spectra and their construction are almost always to [6]. This is for convenience since that is where the results are stated in the form we need. In fact all results about Brown-Gitler spectra at $\mathrm{BP}\langle -1\rangle = H\mathbb{Z}/p$ are due to [1] when $p=2$ and [3] when p is odd. Brown-Gitler spectra at $\mathrm{BP}\langle 0\rangle = H\mathbb{Z}_p^\wedge$ are due to Mahowald [9] when $p=2$ and Kane [7] for p odd and also to Goerss [4] and Shimamoto [10].

1. Acyclic resolutions

In this section we collect all the definitions and results from [6] regarding the construction and realization of the acyclic resolutions that we need. Their work made extensive use of the Λ algebra which we will not do, since their work is sufficient to construct the acyclic resolution we need and the Λ algebra's work is done once they have been constructed. Throughout this section, if the prime you have chosen is $p=2$, replace P^i by Sq^{2i} and β by Sq^1.

For $n \geq 0$, let E_n be the Hopf subalgebra of the Steenrod algebra A, generated by the Milnor elements $Q_0, Q_1, \ldots, Q_n$, let $\bar{E}_n$ be the augmentation ideal and J_n the left A ideal generated by $\bar{E}_n$. Notice that $H^*\mathrm{BP}\langle n\rangle \cong A/J_n$. For $n \geq 1$, let $M_n(k) = A/I_k + J_n$ (where $J_{-1} = \emptyset$). Theorem 2.6 of [6] gives an acyclic resolution of $M_n(k)$ by A modules

$$\cdots \to D_{q+1}^n \xrightarrow{d_q^*} D_q^n \to \cdots \to D_1^n \xrightarrow{d_0^*} D_0^n \xrightarrow{\varepsilon} M_n(k) \to 0$$

for $n=0$ and 1. Their proof works equally well for $n=2$. One should notice that

$D_0^2 = H^*\mathrm{BP}\langle 2\rangle$, $D_1^2 = \bigvee_{j>k} \Sigma^{qj} H^*\mathrm{BP}\langle 1\rangle$ and D_2^2 is a sum of $H^*\mathrm{BP}\langle 0\rangle$'s while for $n \geq 3$, D_n^2 is a free module and a direct summand of D_n^{-1} which is also free. In general, D_m^n is a direct sum of copies of $H^*\mathrm{BP}\langle\max(-1, n-m)\rangle$ and the set indexing the copies is a subset of the set indexing copies of D_m^{n-r} if $r > 0$. Thus there are quotient maps $\theta_q^* : D_q^{-1} \to D_q$ so that

$$\begin{array}{ccc} D_{q+1}^{-1} & \xrightarrow{d_q^*} & D_q^{-1} \\ \Big\downarrow \theta_{q+1}^* & & \Big\downarrow \theta_q^* \\ D_{q+1}^n & \longrightarrow & D_q^n \end{array}$$

commutes and for $q \geq 2$, θ_q^* is a projection onto a direct summand. We can choose spectra K_q^n such that $H^* K_q^n \cong D_q^n$ and get the following:

Proposition 1.1. *For $q \geq 0$, $0 \leq n \leq 2$, there exist maps $\theta_q : K_q^n \to K_q^{-1}$ and $d_q : K_q^n \to \Sigma K_{q+1}^n$ so that*

(i) *$H^*(\theta_q) : D_q^{-1} \to D_q^n$ is just the θ_q^* mentioned above,*

(ii) *$H^*(d_q) : D_{q+1}^n \to D_q^n$ is the differential d_q^* in the acyclic resolution,*

(iii)
$$\begin{array}{ccc} K_q^n & \xrightarrow{d_q} & \Sigma K_{q+1}^n \\ \Big\downarrow \theta_q & & \Big\downarrow \theta_{q+1} \\ K_q^{-1} & \xrightarrow{d_q} & \Sigma K_{q+1}^{-1} \end{array}$$
commutes,

(iv) *if $q \geq n+1$ there is a map $s_q : K_q^{-1} \to K_q^n$ so that $s_q\theta_q$ is the identity.*

Proof. Cases $n = 0$ and 1 were done in Proposition 2.7 of [6]. For $n = 2$, parts (i) and (iv) follow immediately from the structure of the D_q^n's mentioned earlier in this paragraph. Since for $q \geq 2$, K_q^2 is a wedge of Eilenberg-MacLane spectra, the only thing that needs to be done is the construction of d_0 and d_1. To get d_0, we want to realize d_0^*. Looking at d_0's effect on just one factor (see [6]), we have

$$\begin{array}{ccc} \Sigma^{2(p-1)j} H^*\mathrm{BP}\langle 1\rangle \to \bigoplus\limits_{i>k} \Sigma^{2(p-1)i} H^*\mathrm{BP}\langle 1\rangle & \xrightarrow{d_0^*} & H^*\mathrm{BP}\langle 2\rangle \\ a \longmapsto & & a \cdot \chi(P^j). \end{array}$$

This is well defined since the formula

$$P^j Q_i - Q_i P^j = Q_{i+1} P^{j-p^i}$$

implies that $P^j Q_0$ and $P^j Q_1$ are in $\bar{E}_2 \cdot A$ so that $Q_0 \chi P^j$ and $Q_1 \chi P^j$ are zero in $H^*\mathrm{BP}\langle 2\rangle \cong A/A \cdot \bar{E}_2$. Since, as proved in [8], the Adams spectral sequence for

$[\mathrm{BP}\langle 2\rangle, \mathrm{BP}\langle 1\rangle]$ collapses to its E_2 term, this element of $E_2^{0,2(p-1)j} = \mathrm{Hom}^{2(p-1)j}(H^*\mathrm{BP}\langle 1\rangle, H^*\mathrm{BP}\langle 2\rangle)$ gives a map $S^j = \mathrm{BP}\langle 2\rangle \to \Sigma^{2(p-1)j}\,\mathrm{BP}\langle 1\rangle$ making

$$\begin{array}{ccc} \mathrm{BP}\langle 2\rangle & \xrightarrow{S^j} & \Sigma^{2(p-1)j}\mathrm{BP}\langle 1\rangle \\ \downarrow \rho & & \downarrow \rho \\ H\mathbb{Z}/p & \xrightarrow{\chi P^j} & \Sigma^{2(p-1)}H\mathbb{Z}/p \end{array}$$

commute. The map d_1 is handled in the same way. □

2. Adapted complexes

Let B a subalgebra of A and E a ring spectrum with $H^*E = A//B = A/A\bar{B}$. Write $N(k)$ for $A/A\{\bar{B}, w_k\}$ where $w_k = \{\chi P^i \mid i > k\}$ and let $1 : E \to H$ be the generator of H^*E over A. If Z is a finite CW complex and $h' \in E_m Z$ with $m \leq 2p(k+1) - 1$ and $h : \Sigma^m\, DZ \to E$ is the dual of h', then we say that (Z, h') is *adapted to* $N(k)$ if

$$A\{\bar{B}, w_k\} \to A \xrightarrow{h^* \cdot 1^*} H^*\,\Sigma^m\, DZ$$

is exact. The degree of the adapted complex is the degree m of h' in E_*Z. The aim of this section is to prove the following theorem.

Theorem 2.1. *For each $k \geq 0$, there is a finite CW complex Z_k and $h'_k \in \mathrm{BP}\langle 2\rangle_m Z_k$ with $m = 2pk + 3$ so that (Z_k, h'_k) is adapted to $M_2(k)$. Also h'_k can be chosen so that $d_{0*}h'_k = 0$ where d_0 is as in Proposition* 1.1.

Let $R = \{\chi P^I \mid Q_2Q_1Q_0P^I \neq 0,\ I \text{ admissible},\ i_1 \geq k\}$. If $\varepsilon : A \to M_2(k)$ is the projection map, then $\varepsilon(R)$ forms a vector space basis for $M_2(k)$. Lemma 3.6 of [6] reduces the construction of an adapted complex for $M_2(k)$ to proving the next proposition.

Proposition 2.2. *For each $P \in R$ there is a CW complex Z_p and $\bar{h}_P \in H\mathbb{Z}/p_{2pk+3}Z_P$ with*
(1) $\chi(P)^*\bar{h}_P \neq 0$,
(2) *$\bar{h}_P = 1_* h'_P$ for some $h'_P \in \mathrm{BP}\langle 2\rangle_* Z_P$ where 1_* is the reduction*

$$\mathrm{BP}\langle 2\rangle_* Z_P \xrightarrow{1_*} H\mathbb{Z}/p_* Z_P.$$

This will suffice since their lemma tells us that $Z = \bigvee_{P\in R} Z_P$ and $h' = \bigoplus_{P \in R} h'_P$ will be adapted to $M_2(k)$.

Proof of Proposition 2.2. Take any $\chi P^I \in R$, so $Q_2Q_1Q_0P^I \neq 0$, $I = (i_1, \varepsilon_1, i_2, \varepsilon_2, \ldots, i_m, \varepsilon_m)$ is admissible and $i_1 \geq k$. $Q_2Q_1Q_0P^I \neq 0$, implies $\varepsilon_1 = 0 = \varepsilon_2$. Set

$$e = 2i_1 - \sum_{j=2}^{m} 2(p-1)i_j - \sum_{j=3}^{m} \varepsilon_j, \quad t = \sum_{i=3}^{m} \varepsilon_i \quad \text{and} \quad s = \tfrac{1}{2}(e - t).$$

The Z associated to χP^I will be a subcomplex of

$$C(s, t+3) = \bigwedge_s \mathbb{C}P^\infty \wedge \bigwedge_{t+3} B\mathbb{Z}/p,$$

the s-fold smash product of $\mathbb{C}P^\infty$ smashed with the $(t+3)$-fold smash product of $B\mathbb{Z}/p$. Then $H^{2s+t+3}C(s, t+3)$ is the first nonzero group and is generated by a single nonzero element c.

Let pr be the vector space map

$$\text{pr:}\ \ H^*C(s, t+3) \to (H^*C(s, t+3)/\bar{E}_2 \cdot H^*C(s, t+3)).$$

Now s and t have been chosen so that $Q_2Q_1Q_0P^IC \neq 0$. This implies that P^Ic is not in $\bar{E}_2 \cdot H^*C(s, t+3)$ so that $v = \text{pr}(P^Ic) \neq 0$. Dualizing we get

$$(P^I)^*\text{pr}^*(v^*) \neq 0 \quad \text{in } H_*C(s, t+3).$$

Let X_I be a finite skeleton of $C(s, t+3)$ containing the $2p(k+1) + |Q_0| + |Q_1| + |Q_2|$ skeleton and such that $H\mathbb{Z}^\wedge_{p*}X_I$ is an F_p vector space. Then $H\mathbb{Z}^\wedge_{p*}X_I \to H\mathbb{Z}/p_*X_I$ is a monomorphism. Let $j = 2pk - 2pi_1$ and $Z_I = \Sigma^j X_I$. Let $c \in H^{j+2s+t+3}Z_I$ be the generator. Then we still have $\text{pr}(P^Ic) \neq 0$ in $H^*Z_I/\bar{E}_2 \cdot H^*Z_I$. Setting $\bar{h}_I = \text{pr}^*(\text{pr}(P^Ic))^* \in H_{2pk+3}Z_I$ we find that $(P^I)^*\bar{h}_I \neq 0$. All that we have left to show is that $\bar{h}_I$ is in the image of the reduction map $1_*: \text{BP}\langle 2\rangle_*Z_I \to H_*Z_I$. This follows from putting together some facts from [6] and [5]. In [5], Goerss proves that the Adams spectral sequence converging to $\text{BP}\langle 2\rangle_*C(s, t)$ collapses to its E_2 term. This ensures that in the required range of dimensions the Adams spectral sequence converging to $\text{BP}\langle 2\rangle_*Z_I$ collapses to its E_2 term. As pointed out in [6], one easily checks that this E_2 term satisfies

$$\begin{aligned}\text{Ext}^{0,*}_A(H^*\text{BP}\langle 2\rangle \otimes H^*Z_I, \mathbb{Z}/p) &\cong \text{Hom}^*_{E_2}(H^*Z_I, \mathbb{Z}/p)\\ &\cong (H^*Z_I/\bar{E}_2 \cdot H^*Z_I)^*\end{aligned}$$

and the composite.

$$\text{BP}\langle 2\rangle_*Z_I \to E^0_\infty \cong (H^0Z_I/\bar{E}_2 \cdot H^*Z_I)^* \to H\mathbb{Z}/p_*X$$

is the reduction induced by $1: \text{BP}\langle 2\rangle \to H\mathbb{Z}/p$. From this we can conclude that $\bar{h}_I = \text{pr}^*(\text{pr}\ P^Ic)^*$ is in the image of this reduction map; so choose some h'_I with $1_*h'_I = h_I$.

Having constructed the adapted complex mentioned in Theorem 2.1, all that remains to be proved of that theorem is that $d_{0*}h'_I = 0$. The fact that P^Ic generated a free E_2 module in H^*Z_I means that h'_I will generate a submodule of $\text{BP}\langle 2\rangle_*Z_I$ with trivial p, v_1 and v_2 actions. This means that if we split $\text{BP}\langle 2\rangle \wedge Z_I \simeq B \vee \text{KV}$, where KV is the Eilenberg-MacLane spectrum for some graded vector space V, and H^*B has no free summands as a module over A; then $h'_I: S \to B \vee \text{KV}$ is given by $* \vee f$ for some $f: S \to \text{KV}$. Since, for any spectrum X, $[\text{KV}, X] \cong \text{Hom}_A(A_*, H_*X)$

(the appropriate Adams spectral sequence collapses), we notice that the composite $S^j \circ h'_I$ is nonzero if and only if the induced map in homology is. Consider

$$\begin{array}{ccccc} S^m & \xrightarrow{h'_I} & B \vee \mathrm{KV} & \xrightarrow{S^j \vee \mathrm{Id}} & \Sigma^{2(p-1)j}\mathrm{BP}\langle 1\rangle \wedge Z_I \\ & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho \wedge \mathrm{Id}} \\ & & H\mathbb{Z}/p \wedge Z_I & \xrightarrow{\chi P^j \wedge \mathrm{Id}} & \Sigma^{2(p-1)j}H\mathbb{Z}/p \wedge Z_I \end{array}$$

For dimensional reasons, since Z_I is a space, if $j > k$, $\chi P^i \circ \rho \circ h'_I \cong *$. Since reduction from BP⟨1⟩ to $H\mathbb{Z}/p$ is a monomorphism in homology, we find that $S^i \circ h'_I$ is zero in homology. Hence $S^j \circ h'_I \simeq *$ when $j > k$ so $d_{0*}h'_I = 0$.

3. Construction of BP⟨2⟩^{k}

In this section we prove Theorem A for $n = 2$. The proof uses the fact that it has already been proved for $n = -1$. To get started we need the following lemma which encapsulates the role adaptive complexes play in the construction.

Lemma 3.1. *Let* (T, h') *be the adapted complex constructed in Section* 2 *and* $h : DT \to K_0^2 = \mathrm{BP}\langle 2\rangle = X_0$ *the dual of* h'. *Given parts* (a), (b) *and* (c) *of Theorem A for* $q \leq t$, *i.e. cofiber sequences*

$$K_{q+1}^2 \xrightarrow{i_{q+1}} X_{q+1} \xrightarrow{p_q} X_q \xrightarrow{e_q} \Sigma K_{q+1}^2$$

with $e_q i_q = d_q$, *and given liftings* $h_q : \mathrm{DT} \to X_q$ *for* $q \leq t+1$. *Then* $\ker i_q^* \cap \ker h_q^* = 0$ *for* $q \leq t$.

Proof. The proof is by induction on q. There are two base cases. If $q = 0$ the result follows since $i_0 = \mathrm{Id}$. If $q = 1$, take $v \in \ker i_1^*$. Then $v = p_1^* w$ for some $w \in H^*X_0$. Now $\ker p_1^* = \mathrm{Im}\, e_0^*$ implies that either $v = 0$ in which case we are done or else $w \notin \mathrm{Im}\, e_0^* = \mathrm{Im}\, d_0^* = \ker h_0^*$. The last equality is a consequence of Theorem 2.1. Thus $0 \neq h_0^* w = h_1 p_1^* w = h_1^* v$ where the second equality is a consequence of the fact that h_1 is a lifting of h_0. Hence $\ker i_1^* \cap \ker h_1^* = 0$.

Now assume inductively that $\ker i_q^* \cap \ker h_q^* = 0$ for $q \leq s < t$. Take $v \in \ker i_{s+1}^*$. Then $v = p_s^* w$ for some $w \in H^*X_s$. Now $i_s^* w \in \ker d_{s-1}^* = \mathrm{Im}\, d_s^*$ so choose $x \in H^*K_{s+1}^2$ such that $d_s^* x = i_s^* w$. Let $w' = w - e_s^* x$. Then $w' \in \ker i_s^*$ so by our induction hypothesis $h_s^* w' \neq 0$. But

$$h_s^* w' = h_{s+1}^* p_s^*(w - e_s^* x) = h_{s+1}^* v.$$

So $\ker i_{s+1}^* \wedge \ker h_{s+1}^* = 0$. □

Proof of Theorem A for $n = 2$. Let

$$\cdots \to Y_{q-1} \xrightarrow{p_{q-1}} Y_{q-1} \to \cdots \to Y_1 \to Y_0 = H\mathbb{Z}/p$$

be the tower whose homotopy inverse limit is $B(k)$, i.e. use $n=-1$ of Theorem A. We have the cofiber sequences

$$K_{q+1}^{-1} \xrightarrow{i_{q+1}} Y_{q+1} \xrightarrow{p_q} Y_q \xrightarrow{e_q} K_{q+1}^{-1}$$

and maps with the properties of Theorem A. One additional property of this case which we heavily depend on is given as Lemma 1.9 in [6], namely: for Z a finite complex, $m \leq 2p(k+1)-1$ and $\gamma: \Sigma^m DZ \to H\mathbb{Z}/p = Y_0$, any lifting of γ to Y_q lifts to Y_{q+1}. Let (Z, h') be the complex adapted to $M_2(k)$ constructed in Section 2. Set $T = \Sigma^{2pk+3}DZ$ and let $h: T \to \mathrm{BP}\langle 2\rangle$ be dual to h'.

We proceed by induction on q with the hypothesis:

$H(t)$: For $q \leq t+1$, there are spectra X_q and maps e_q, i_{q+1} for $q \leq t$ so that properties (a)-(d) of Theorem A hold. Additionally, there are maps $\theta'_q: X_q \to Y_q$ for $q \leq t+1$ so that for $q \leq t$

$$\begin{tikzcd}
K_{q+1}^{2} \arrow[r, "i_{q+1}"] \arrow[d, "\theta_{q+1}"] & X_{q+1} \arrow[r, "p_q"] \arrow[d, "\theta'_{q+1}"] & X_q \arrow[r, "e_q"] \arrow[d, "\theta'_q"] & \Sigma K_{q+1}^{2} \arrow[d, "\theta_{q+1}"] \\
K_{q+1}^{-1} \arrow[r, "i_{q+1}"] & Y_{q+1} \arrow[r, "p_q"] & Y_q \arrow[r, "e_q"] & \Sigma K_{q+1}^{-1}
\end{tikzcd} \tag{3.2}$$

is a commutative diagram of cofibration sequences. Also, h lifts to X_{t+1}.

Now X_0 is BP⟨2⟩ and e_0 is $d_0: \mathrm{BP}\langle 2\rangle \to \bigvee_{j>k} \Sigma^{2(p-1)j}\mathrm{BP}\langle 1\rangle$. X_1 is defined to be the fiber of e_0 and setting i_0 to be the identity gives $e_0 i_0 = d_0$. $\theta'_0: \mathrm{BP}\langle 2\rangle \to H\mathbb{Z}/p$ is the usual reduction map and θ'_1 can be chosen to be any map satisfying (3.2), which completes the proof of $H(0)$.

$H(1)$ needs to be dealt with in a special manner. Define e_1 to fill in the following diagram of cofibrations, where the top square commutes because $d_1 \circ d_0 = 0$.

$$\begin{tikzcd}
\Sigma^{-1}\mathrm{BP}\langle 2\rangle = \Sigma^{-1}K_0^2 \arrow[r] \arrow[d, "d_0"] & * \arrow[d] \\
K_1^2 \arrow[r, "d_1"] \arrow[d, "i_1"] & \Sigma K_2^2 \arrow[d, dash] \\
X_1 \arrow[r, dashed, "e_1"] & \Sigma K_2^2
\end{tikzcd}$$

So by definition, $e_1 i_1 = d_1$. To get $e_1\theta'_1 = \theta_2 e_1$ we may have to adjust e_1 a bit. Setting $\Delta = e_1\theta'_1 - \theta_2 e_1$ we find that

$$\Delta i_1 = e_1\theta'_1 - \theta_2 d_1 = d_1\theta_1 - \theta_2 d_1 = 0.$$

Thus the top square in the following diagram commutes defining g.

$$\begin{tikzcd} K_1^2 \arrow[r] \arrow[d, "i_1"] & * \arrow[d] \\ X_1 \arrow[r, "e_1\theta_1'-\theta_2 e_1"] \arrow[d, "p_0"] & K_2^{-1} \arrow[d, equal] \\ \mathrm{BP}\langle 2\rangle \arrow[r, dashed] & K_2^{-1} \end{tikzcd}$$

If $\rho:\mathrm{BP}\langle 2\rangle \to H$ is the reduction map, then, since $\rho^*: H^*H \to H^*\mathrm{BP}\langle 2\rangle$ is surjective and K_2^{-1} is a generalized Eilenberg-MacLane spectrum, g factors as

$$\mathrm{BP}\langle 2\rangle \xrightarrow{\rho=\theta_0} H = Y_0 \xrightarrow{g'} K_2^{-1}.$$

Define $e_1' = e_1 - g'p_0: Y_1 \to K_2^{-1}$. Notice that in the tower defining $B(k)$ given in Theorem A, e_1 can be replaced by e_1' since

$$e_1' i_1 = (e_1 - g'p_0)i_1 = e_1 i_1 = d_1.$$

This replacement may require adjustment of the higher e_i's but this can be done as in the original proof. So

$$e_1'\theta_1' - \theta_2 e_1 = \Delta - g'p_0\theta_1 = \Delta - g'\theta_0' p_0 = \Delta - \Delta = 0.$$

To get the lifting of h_1 notice that $\theta_1' h_1$ lifts, so $\theta_2 e_1 h_1 = e_1\theta_1' h_1 = 0$. T was chosen so that $\mathrm{BP}\langle 0\rangle^* T \to H\mathbb{Z}/p^* T$ is a monomorphism, hence $e_1 h_1 = 0$ and h_1 lifts to $h_2: T \to X_2$, completing the verification of $H(2)$.

Now assume $H(t-1)$ with $t \geqslant 2$. This proof of the induction step is essentially that of [6]. Since $t+1 \geqslant 3$, there is a map $s_{t+1}: K_{t+1}^{-1} \to K_{t+1}^2$ such that $s_{t+1}\theta_{t+1}$ is the identity. Set

$$e_t = s_{t+1} e_t \theta_t': \ X_t \to \Sigma K_{t+1}^2$$

and let X_{t+1} be the fiber of e_t. Then by our induction hypothesis,

$$e_t i_t = s_{t+1} e_t \theta_t' i_t = s_{t+1} e_t i_t \theta_t = s_{t+1} d_t \theta_t = s_{t+1}\theta_{t+1} d_t = d_t.$$

To lift h_t to X_{t+1}, notice that $\theta_t' h_t$ is a lifting of $\theta_0' h$. By Lemma 1.7 of [6] this must lift even further so we must have $e_t\theta_t' h_t = 0$ so $0 = s_{t+1} e_t \theta_t' h_t = s_{t+1}\theta_{t+1} e_t h_t = e_t h_t$ and h_t lifts. To get the diagram (3.2) we need to show

$$\begin{tikzcd} X_t \arrow[r, "e_t"] \arrow[d, "\theta_t'"] & \Sigma K_{t+1}^2 \arrow[d, "\theta_{t+1}"] \\ Y_t \arrow[r] & \Sigma K_{t+1}^{-1} \end{tikzcd}$$

commutes and then simply define θ_{t+1}' to satisfy (3.2). Since K_{t+1}^{-1} is an Eilenberg-MacLane spectrum, it suffices to show $\theta_{t+1} e_t = e_t \theta_t'$ in cohomology. Lemma 3.1

indicates that it is sufficient to show $i_t^* e_t^* \theta_{t+1}^* = i_t^* \theta_t'^* e_t^*$ and $h_t^* e_t^* \theta_{t+1}^* = h_t^* \theta_t'^* e_t^*$. However

$$\theta_{t+1} e_t i_t - e_t \theta_t' i_t = \theta_{t+1} d_t - e_t i_t \theta_t = \theta_{t+1} d_t - d_t \theta_t = 0$$

and an earlier argument showed $e_t h_t = 0 = e_t \theta_t' h_t$. Finally, property (d) of Theorem A holds here because

$$(\theta_{t+1})_*:\ (K_{t+1}^2)_* Z \ \rightarrow\ (K_{t+1})_* Z$$

is injective. This completes the proof of the inductive step and the theorem. □

To finish this section, we need to complete the proof of Theorem B begun in the introduction. Recall that $\mathrm{BP}\langle 2\rangle^k$ is defined to be $\mathrm{holim}_q X_q$. We have already shown that if Z is a spectrum, $w: \mathrm{BP}\langle 2\rangle^k \to \mathrm{BP}\langle 2\rangle$ the map from $\mathrm{holim}_q X_q$ to X_0 and $m \leq 2p(k+1)-1$ then $\{\mathrm{Im}\ w_*: \mathrm{BP}\langle 2\rangle_m^k Z \to \mathrm{BP}\langle 2\rangle_m Z\} \subseteq \bigcap_{j>k} \ker S_*^j$ and we need to show that if $H\mathbb{Z}_p^\wedge{}_* Z$ is an F_p vector space then $\mathrm{Im}\ w_* = \bigcap_{j>k} \ker S_*^j$.

Proof. Smash all of the diagrams used to construct $\mathrm{BP}\langle 2\rangle^k$ with Z. Take any $f: S^m \to \mathrm{BP}\langle 2\rangle \wedge Z$ with the property that $f \in \cap_{j>k} \ker S_*^j \subseteq \mathrm{BP}\langle 2\rangle_m Z$ and $m \leq 2p(k+1)-1$. The first condition on f guarantees that f lifts to $f_1: S^m \to X_1 \wedge Z$. Let $\bar{f}_1$ be the composite

$$S^m \xrightarrow{f_1} X_1 \wedge Z \xrightarrow{\theta_1' \wedge \mathrm{Id}} Y_1 \wedge Z.$$

Then by Lemma 1.9 of [6], $\bar{f}_1$ lifts to $\bar{f}_2: S^m \to Y_2 \wedge Z$ so that $(e_1 \wedge \mathrm{Id}_Z) \circ \bar{f}_1 = 0$. Writing Z to denote the identity map on Z, the commutativity of (3.2) tells us that

$$0 = (e_1 \wedge Z) \circ (\theta_1' \wedge Z) \circ f_1 = (\theta_2 \wedge Z) \circ (e_1 \wedge Z) \circ f_1.$$

Since $H\mathbb{Z}_p^\wedge{}_* Z$ is an F_p vector space, this means that $(e_1 \wedge Z) \circ f_1 = 0$ and so f_1 lifts to f_2. A similar argument shows that f_2 and higher lifts can always be lifted. These arguments don't need $H\mathbb{Z}_p^\wedge{}_* Z$ an F_p vector space, since K_s^2 is a wedge of $H\mathbb{Z}/p$'s for $s \geq 3$, but do use the fact that K_s^2 is a wedge summand of K_s^{-1} for $s \geq 3$. Thus f lifts all the way up the tower to give $f': S^m \to \mathrm{BP}\langle 2\rangle^k \wedge Z$ with $w \circ f' = f$.

4. Spacelike spectra and maps to Brown–Gitler spectra

Proof of Corollary D. Throughout this proof, all pairs (s, t) should be assumed to satisfy $s - t \geq n - 2p(k+1) + 2$. The first thing to notice is that since Y is a finite spectrum, $[Y, \mathrm{BP}\langle 2\rangle^k] \cong [s, DY \wedge \mathrm{BP}\langle 2\rangle^k]$, and we can replace $E_r^{**}(Y, \mathrm{BP}\langle 2\rangle^k)$ by $E_r^{**}(DY \wedge \mathrm{BP}\langle 2\rangle^k)$. Let $f: \Sigma^n DZ \to Y$ be the map making Y spacelike. Then $Df: DY \to \Sigma^{-n} Z$ has the property that $Df_*: H_* DY \to H_* \Sigma^{-n} Z$ is a monomorphism. Df induces a map $i: E_1^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \to E_1^{s,t}(\Sigma^{-n} Z \wedge \mathrm{BP}\langle 2\rangle^k)$ which is monic for $s \geq 3$ since Df_* is monic. Z being a space, Corollary C tells us that all differentials in $E_r^{**}(\Sigma^{-n} Z \wedge \mathrm{BP}\langle 2\rangle^k)$ are zero except possibly for d_1 and d_2.

The proof is by induction on r in the assertion that for $s-t \geqslant n-2(k+1)+2$,

$$E_3^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \cong E_r^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \text{ for all } r \geqslant 3 \tag{A}$$

and

$$i: \; E_r^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \to E_r^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k) \quad \text{is monic for } s \geqslant 3. \tag{B}$$

The base case is $r=3$ where (A) is immediate. As noted above, $i: E_1^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \to E_1^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k)$ is monic for $s \geqslant 3$. Now consider the first differential. In $E_1^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle)$, d_1 is zero on $E_1^{s,t}$ for $s \geqslant 2$ because in that range K_{s+1}^2 is a wedge summand of K_{s+1}^{-1}, and maps to the original Brown-Gitler tower always lift. Said more explicitly, $x \in E_1^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k)$ is a map $x: S^{s-t-n} \to \Sigma^{-n}Z \wedge K_s^2$.

$$\theta_{s+1}d_1x = \theta_{s+1}e_si_sx = e_{s+1}\theta'_{s+1}i_sx = 0$$

because $\theta'_{s+1}i_sx = S^{s-t-n} \to \Sigma^{-n}Z \wedge Y_{s+1}$ lifts. However $s+1 \geqslant 3$ implies that θ_{s+1} is an inclusion of a wedge summand so $d_1x = 0$. Thus

$$E_1^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k) \cong E_2^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k).$$

Therefore d_1 is zero on $E_1^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k)$ for $s \geqslant 3$ since $0 = d_1ix = \mathrm{id}_1x$ and i is monic. A similar argument shows d_2 is zero on $E_2^{s,t}$ for $s \geqslant 1$ so

$$E_3^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k) \cong E_2^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k) \quad \text{for } s \geqslant 3 \text{ and}$$

$$E_3^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \cong E_2^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k) \quad \text{for } s \geqslant 5 \text{ and}$$

$$i: \; E_3^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k \to E_3^{s,t}(\Sigma^{-n}Z \wedge \mathrm{BP}\langle 2\rangle^k) \quad \text{is monic for } s \geqslant 3.$$

For the induction step, assume both are true for some $n \geqslant 3$. Since d_nx has filtration $\geqslant 3$ for any $x \in E_n^{s,t}(DY \wedge \mathrm{BP}\langle 2\rangle^k)$, $0 = d_nix = id_nx$. Since i is monic in this range, $d_nx = 0$. Thus both $E_{n+1}^{s,t}$ terms are isomorphic to their respective $E_n^{s,t}$ terms and i remains monic on $E_{n+1}^{s,t}$ for $s \geqslant 3$. This completes the induction step. □

One final comment is in order. [6] used these spectral sequences to construct pairings $\mathrm{BP}\langle 1\rangle^i \wedge \mathrm{BP}\langle 1\rangle^j \to \mathrm{BP}\langle 1\rangle^{i+j}$ and to prove the uniqueness of $\mathrm{BP}\langle 1\rangle^k$. I have not been able to prove the analogous results for the $\mathrm{BP}\langle 2\rangle^k$ case because of the extra differential involved.

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